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Higher order potential expansion for the continuous limits of the Toda hierarchy

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Abstract

A method for introducing the higher order terms in the potential expansion to study the continuous limits of the Toda hierarchy is proposed in this paper. The method ensures that the higher order terms are differential polynomials of the lower ones and can be continued to be performed indefinitely. By introducing the higher order terms, the fewer equations in the Toda hierarchy are needed in the so-called recombination method to recover the KdV hierarchy. It is shown that the Lax pairs, the Poisson tensors and the Hamiltonians of the Toda hierarchy tend towards the corresponding ones of the KdV hierarchy in the continuous limit.

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1. Introduction

The continuous limits of discrete systems are one of the remarkably important research areas in soliton theory [1–4]. In recent years, more attention was focused on the continuous limit relations between hierarchies of discrete systems and hierarchies of soliton equations [5–9]. The so-called recombination method, i.e. properly combining the objects (such as the vector fields) of discrete systems, was first proposed to study the continuous limit of the Ablowitz–Ladik hierarchy [5] and the Kac–van Moerbeke hierarchy [6]. Morosi and Pizzocchero also used the recombination method to study the continuous limits of some integrable lattices in their recent works [7–9]. Up to now, there has not been much work concerning the continuous limit relations between lattices and differential equations, which have different numbers of potentials. Furthermore, to the best of our knowledge, there is no work which successfully gives a way to introduce the higher order terms in potential expansions to study the continuous limit relations between hierarchies of lattices and hierarchies of soliton equations. Illumined by Gieseker's conjecture [10], we will propose a method for finding the higher order terms in potential expansions to study the continuous limit relation between the Toda hierarchy and the KdV hierarchy by the recombination method.

In 1996, Gieseker proposed a conjecture [10]:

Conjecture. *Denote $w(n,t)$ and $v(n,t)$, where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, as the two potentials of the Toda hierarchy, and let f be a function of $x \in \mathbb{R}$ and $t \in \mathbb{R}$. There are $\Phi_i(f)$, which are the differential polynomials of f, so that if we define*

$$w(n,t) = -2 + f(x,t)h^2 + h^2 \sum_{i=1}^{L} \Phi_i(f(x,t))h^i \tag{1.1a}$$

$$v(n,t) = 1 + f(x,t)h^2 - h^2 \sum_{i=1}^{L} \Phi_i(f(x,t))h^i \tag{1.1b}$$

where h is the small step of lattice and $x = nh$, then by taking suitable linear combinations of the equations of Toda hierarchy under definition (1.1), we can produce asymptotic series whose leading terms in h are the KdV hierarchy if L is large enough.

In [10], Gieseker proposed a way to introduce $\Phi_i(f)$ by using the Toda lattice

$$w_t = v - Ev = v - v^{(1)} \qquad v_t = v\left(E^{(-1)}w - w\right) = v\left(w^{(-1)} - w\right) \tag{1.2}$$

where the shift operator E is defined by

$$(Ef)(n) = f(n+1) \qquad f^{(k)}(n) = E^{(k)}f(n) = f(n+k) \qquad n, k \in \mathbb{Z}.$$

For instance, in order to find $\Phi_1(f)$, substituting definition (1.1) into equation (1.2) and expanding the shift terms by Taylor's theorem

$$\frac{\mathrm{d}f}{\mathrm{d}t} + \frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}t}h = -\frac{\mathrm{d}f}{\mathrm{d}x}h - \frac{\mathrm{d}^2 f}{2\,\mathrm{d}x^2}h^2 + \frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}x}h^2 + O(h^3) \tag{1.3a}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} - \frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}t}h = -\frac{\mathrm{d}f}{\mathrm{d}x}h + \frac{\mathrm{d}^2 f}{2\,\mathrm{d}x^2}h^2 - \frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}x}h^2 + O(h^3). \tag{1.3b}$$

Combining the above two equations we know

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{\mathrm{d}f}{\mathrm{d}x}h + O(h^3) \tag{1.4}$$

then by the chain rule we have

$$\frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}t} = -\frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}x}h + O(h^2). \tag{1.5}$$

Note from the above equation and equation (1.3*a*) one can get

$$\frac{\mathrm{d}\Phi_1(f)}{\mathrm{d}x} = \frac{1}{4}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \tag{1.6}$$

which by integration yields

$$\Phi_1(f) = \frac{1}{4}\frac{\mathrm{d}f}{\mathrm{d}x}. \tag{1.7}$$

We can see that the integration must be used in this process for finding $\Phi_i(f)$. As a consequence, there is a problem whether this process can be continued indefinitely and the $\Phi_i(f)$, found in this process, are the differential polynomials of f.

Gieseker's conjecture was proved in the following three cases of (1.1) [11]:

(a) $L = 0$, $f(x,t) = \frac{1}{2}q(x,t)$;
(b) $L = 1$, $f(x,t) = \frac{1}{2}q(x,t)$, $\Phi_1(f) = \frac{1}{8}q_x$;
(c) $L = 2$, $f(x,t) = \frac{1}{2}q(x,t)$, $\Phi_1(f) = \frac{1}{8}q_x$, $\Phi_2(q) = -\frac{1}{32}q^2$.

It was found that fewer equations in the Toda hierarchy are needed in the recombination method for case (c) to give the KdV hierarchy than for case (a).

In this paper, we will give a new method to introduce $\Phi_i(f)$ required in (1.1) instead of Gieseker's process so that we can derive the continuous limit relation between the Toda hierarchy and the KdV hierarchy by the recombination method. Following our approach for finding $\Phi_i(f)$, one can easily see that the $\Phi_i(f)$ are all differential polynomials of f. Compared with the previous work in [11], we will show that fewer equations in the Toda hierarchy are needed in the recombination method for giving the KdV hierarchy if higher order terms are introduced in the potential expansion (1.1). We will also show that the Lax pairs, the Poisson tensors and the Hamiltonians of the Toda hierarchy tend towards the corresponding ones of the KdV hierarchy in the continuous limit.

2. Basic notation and some known results

For later use, we list some notation and results in [11]. Let us consider the following discrete isospectral problem [12, 13],

$$Ly = \left(E + w + vE^{(-1)}\right) y = \lambda y \tag{2.8}$$

where $w = w(n,t)$ and $v = v(n,t)$ depend on integer $n \in \mathbb{Z}$ and real variable $t \in \mathbb{R}$, and λ is the spectral parameter.

The equation in the Toda hierarchy associated with (2.8) can be written as the following Hamiltonian equation [12]:

$$\begin{pmatrix} w \\ v \end{pmatrix}_{t_m} = JK_{m+1} = J\frac{\delta H_{m+1}}{\delta u} \qquad m = 0, 1, \ldots \tag{2.9}$$

where $\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta w}, \frac{\delta}{\delta v}\right)^T$, and the Poisson tensor J and the Hamiltonians H_i are defined by

$$J \equiv \begin{pmatrix} 0 & J_{12} \\ J_{21} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & (1-E)v \\ v\left(E^{(-1)} - 1\right) & 0 \end{pmatrix}$$

$$K_i \equiv \begin{pmatrix} K_{i,1} \\ K_{i,2} \end{pmatrix} = \frac{\delta H_i}{\delta u} = \begin{pmatrix} -b_i^{(1)} \\ \frac{a_i}{v} \end{pmatrix} \qquad i = 0, 1 \ldots \tag{2.10}$$

$$H_0 = \frac{1}{2}\ln v \qquad H_i = -\frac{b_{i+1}}{i} \qquad i = 1, 2, \ldots$$

with $a_0 = \frac{1}{2}$, $b_0 = 0$ and

$$b_{i+1}^{(1)} = wb_i^{(1)} - \left(a_i^{(1)} + a_i\right) \qquad a_{i+1}^{(1)} - a_{i+1} = w\left(a_i^{(1)} - a_i\right) + vb_i - v^{(1)}b_i^{(2)} \tag{2.11}$$

for $i = 0, 1, \ldots$. The Lax pairs for the mth equation of the Toda hierarchy (2.9) are given by (2.8) and

$$y_{t_m} = A_m y = \sum_{i=0}^{m}\left(-vb_i^{(1)}E^{(-1)} - a_i\right)\left(E + w + vE^{(-1)}\right)^{m-i} y \qquad m = 0, 1, \ldots. \tag{2.12}$$

Equations (2.9) have the bi-Hamiltonian formulation

$$GK_{i-1} = JK_i \qquad i = 1, 2, \ldots$$
$$G \equiv \begin{pmatrix} vE^{(-1)} - v^{(1)}E & w(1-E)v \\ v\left(E^{(-1)} - 1\right)w & v\left(E^{(-1)} - E\right)v \end{pmatrix} \tag{2.13}$$

where G is the second Poisson tensor. The Toda hierarchy also has a tri-Hamiltonian formulation and a Virasoro algebra of master symmetries [14, 15]. The first four covariants K_i are

$$K_0 = \begin{pmatrix} 0 \\ \frac{1}{2v} \end{pmatrix} \qquad K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} w \\ 1 \end{pmatrix} \qquad K_3 = \begin{pmatrix} v + v^{(1)} + w^2 \\ w + w^{(-1)} \end{pmatrix}. \tag{2.14}$$

The Schrödinger spectral problem is given by

$$\bar{L}\bar{y} = \left(\partial_x^2 + q\right)\bar{y} = -\bar{\lambda}\bar{y} \tag{2.15}$$

which is associated with the KdV hierarchy [13]

$$q_{t_m} = B_0 P_m = B_0 \frac{\delta \bar{H}_m}{\delta q} \qquad m = 0, 1, \ldots \tag{2.16}$$

where the vector field possesses the bi-Hamiltonian formulation with two Poisson tensors B_0 and B_1

$$B_0 P_{k+1} = B_1 P_k \qquad k = 0, 1, \ldots$$
$$B_0 = \partial \equiv \partial_x \qquad B_1 = \frac{1}{4}\partial^3 + q\partial + \frac{1}{2}q_x \qquad \bar{H}_i = \frac{4\bar{b}_{i+2}}{2i+1} \qquad i = 0, 1, \ldots \tag{2.17}$$

with $\bar{b}_0 = 0$, $\bar{b}_1 = 1$ and

$$\bar{b}_{i+1} = \left(\tfrac{1}{4}\partial^2 + q - \tfrac{1}{2}\partial^{-1}q_x\right)\bar{b}_i \qquad i = 0, 1, \ldots$$

where $\partial^{-1}\partial = \partial\partial^{-1} = 1$. The first three covariants P_k read as

$$P_0 = 2 \qquad P_1 = q \qquad P_2 = \tfrac{1}{4}\left(3q^2 + q_{xx}\right). \tag{2.18}$$

The well-known KdV equation is the second one:

$$q_{t_2} = \tfrac{1}{4}\left(3q^2 + q_{xx}\right)_x. \tag{2.19}$$

The Lax pairs for the mth equation of the KdV hierarchy (2.16) are given by (2.15) and

$$\bar{y}_{t_m} = \bar{A}_m \bar{y} = \sum_{i=0}^{m} \left(-\frac{1}{2}\bar{b}_{i,x} + \bar{b}_i \partial\right)(\partial^2 + q)^{m-i}\bar{y} \qquad m = 0, 1, \ldots. \tag{2.20}$$

Let us consider the Toda hierarchy on a lattice with a small step h. We interpolate the sequences $(w(n))$ and $(v(n))$ with two smooth functions of a continuous variable x, and relate $w(n)$ and $v(n)$ to $f(x)$ by using (1.1). Suppose

$$E^{(k)}w(n) = -2 + f(x+kh)h^2 + h^2 \sum_{i=1}^{L} \Phi_i(f(x+kh))h^i$$
$$E^{(k)}v(n) = 1 + f(x+kh)h^2 - h^2 \sum_{i=1}^{L} \Phi_i(f(x+kh))h^i \qquad k \in \mathbb{Z}.$$

In [11], we got the following result.

Proposition 1. *Under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$, the Lax operator of the Toda hierarchy goes to the Lax operator of the KdV hierarchy in the continuous limit, i.e. we have*

$$L = \bar{L}h^2 + O(h^3). \tag{2.21}$$

Lemma 1. *Under relation (1.1), we have*

$$K_i = \begin{pmatrix} -b_i^{(1)} \\ \frac{a_i}{v} \end{pmatrix} = \begin{pmatrix} \alpha_i \\ \gamma_i \end{pmatrix} + O(h) \qquad i = 0, 1, \ldots \tag{2.22}$$

where α_i and γ_i are given by

$$\alpha_0 = 0 \qquad \alpha_1 = 1 \qquad \gamma_0 = \tfrac{1}{2} \qquad \gamma_1 = 0 \tag{2.23a}$$

$$\alpha_i = (-1)^{(i-1)} C_{2i-2}^{i-1} \qquad \gamma_i = (-1)^i C_{2i-2}^i \qquad i = 2, 3, \ldots. \tag{2.23b}$$

Define $\tilde{J} = \begin{pmatrix} 0 & \tilde{J}_{21} \\ \tilde{J}_{12} & 0 \end{pmatrix}$ by requiring that $J\tilde{J} = I$. Then the following lemma is true.

Lemma 2. *Under relation (1.1), we have*

$$TK_i \equiv \tilde{J}GK_i = K_{i+1} + \delta_{i+1}K_0 \qquad i = 0, 1, \ldots \tag{2.24}$$

where

$$\delta_i = -2(\alpha_i + \gamma_i) = (-1)^i \frac{2}{i} C_{2i-2}^{i-1} \qquad i = 1, 2, \ldots. \tag{2.25}$$

Proposition 2. *Under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$, the Poisson tensors of the Toda hierarchy go to those of the KdV hierarchy in the continuous limit,*

$$J = -B_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} h + O(h^2) \qquad W_{ij} + W_{kl} = -B_1 h^3 + O(h^4) \tag{2.26}$$

where $W \equiv \frac{1}{4}G\tilde{J}G + G = (W_{ij})$, $1 \leqslant i, j \leqslant 2$, and

$$(i, j, k, l) \in \{(1,1,1,2), (1,1,2,1), (1,2,2,2), (2,1,2,2)\}.$$

3. Higher order potential expansion and the continuous limits of the Toda hierarchy

Now, we give a new method to introduce $\Phi_i(f)$ as required in (1.1) and derive the continuous limits of the Toda hierarchy under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$.

Lemma 3. *Define the operator as*

$$T \equiv \tilde{J}G = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{3.27}$$

Then under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$, the operator T has the following expansions for its entries:

$$T_{11} = -2 + \tfrac{1}{2}h^2 q + O(h^3) \qquad T_{12} = 2 + h\partial + \left(\tfrac{1}{2}\partial^2 + q\right)h^2 + O(h^3)$$

$$T_{21} = 2 - h\partial + \left(\tfrac{1}{2}\partial^2 - \tfrac{1}{2}\partial^{-1}q_x\right)h^2 + O(h^3) \qquad T_{22} = -2 + \tfrac{1}{2}h^2\partial^{-1}q\partial + O(h^3).$$

Proof. The result can be found in [11] (see the proof of lemma 3 in [11]). □

Lemma 4. *Under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$, we have the following expansions,*

$$K_i \equiv \begin{pmatrix} K_{i,1} \\ K_{i,2} \end{pmatrix} = \begin{pmatrix} \alpha_i + \Psi_{i,1,0}(q)h^2 + h^2 \sum_{j=1}^{L} h^j(\zeta_{i,1}\Phi_j + \Psi_{i,1,j}(q,\Phi_1,\ldots,\Phi_{j-1})) \\ \gamma_i + \Psi_{i,2,0}(q)h^2 + h^2 \sum_{j=1}^{L} h^j(\zeta_{i,2}\Phi_j + \Psi_{i,2,j}(q,\Phi_1,\ldots,\Phi_{j-1})) \end{pmatrix} + O(h^{L+3}) \quad (3.28)$$

for $i = 0, 1, 2, \ldots$, where α_i and γ_i are given in lemma 1,

$$\zeta_{0,1} = 0 \qquad \zeta_{0,2} = \tfrac{1}{2} \qquad \zeta_{1,1} = 0 \qquad \zeta_{1,2} = 0$$

$$\zeta_{i+1,1} = -2\zeta_{i,1} + 2\zeta_{i,2} + \alpha_i - 2\gamma_i \qquad \zeta_{i+1,2} = 2\zeta_{i,1} - 2\zeta_{i,2} + \alpha_i - \tfrac{1}{2}\delta_{i+1} \qquad i = 0, 1, \ldots. \quad (3.29)$$

$\Psi_{i,1,j}(q,\Phi_1,\ldots,\Phi_{j-1})$ stands for the term which is a differential polynomial of q, $\Phi_1, \ldots, \Phi_{j-1}$, etc.

Proof. Define $c_i = -vb_i^{(1)}$, $i = 0, 1, \ldots$. Using the identity [12]

$$\sum_{i=0}^{k}(a_i a_{k-i} + b_i c_{k-i}) = 0 \qquad k = 1, 2, \ldots$$

we can show by mathematical induction that a_i, b_i, c_i, $i = 0, 1, \ldots$, are polynomials of $w, v, w^{(-1)}, v^{(-1)}, w^{(1)}, v^{(1)}, \ldots$. According to the definition of K_i in (2.10), we conclude that K_i has the expansion formula (3.28). Note lemmas 1 and 2, we can prove (3.29) by mathematical induction. □

Lemma 5. *Define the combination coefficients $\beta_{k,i}$, $0 \leqslant i \leqslant k+1$, $k = 0, 1, \ldots$, as follows:*

$$\beta_{0,0} = 2 \qquad \beta_{0,1} = 1 \qquad \beta_{1,0} = -2 \qquad \beta_{1,1} = 2 \qquad \beta_{1,2} = 1$$
$$\beta_{k+1,i} = \beta_{k,i-1} \quad 1 \leqslant i \leqslant k+2 \qquad \beta_{k+1,0} = \sum_{i=0}^{k+1} \beta_{k,i}\delta_{i+1} \quad (3.30)$$

then we have

$$\sum_{i=0}^{k+1} \beta_{k,i}\alpha_i = 0 \qquad \sum_{i=0}^{k+1} \beta_{k,i}\gamma_i = 0 \qquad k = 1, 2, \ldots.$$

Proof. It is easy to check the case when $k = 1$. If the lemma is true for k, then

$$\sum_{i=0}^{k+1} \beta_{k,i} K_i = O(h) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so according to lemma 2, we have

$$\sum_{i=0}^{k+2} \beta_{k+1,i} K_i = \tilde{J} G \sum_{i=0}^{k+1} \beta_{k,i} K_i = O(h) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which completes the proof. □

Lemma 6. *Let $\beta_{k,i}$ be defined by (3.30). Then we have*

$$\sum_{i=0}^{k+1} \beta_{k,i}(\zeta_{i,2} - \zeta_{i,1}) = (-4)^k \qquad k = 1, 2, \ldots. \quad (3.31)$$

Proof. It is easy to check the case when $k = 1$. If the lemma is true for k, then we have (according to lemmas 1 and 4)

$$\begin{aligned}\sum_{i=0}^{k+2}\beta_{k+1,i}(\zeta_{i,2}-\zeta_{i,1}) &= \frac{1}{2}\sum_{i=0}^{k+1}\beta_{k,i}\delta_{i+1}+\sum_{i=1}^{k+2}\beta_{k,i-1}(\zeta_{i,2}-\zeta_{i,1})\\ &= \frac{1}{2}\sum_{i=0}^{k+1}\beta_{k,i}\delta_{i+1}+\sum_{i=0}^{k+1}\beta_{k,i}\left(-4\zeta_{i,2}+4\zeta_{i,1}-\frac{1}{2}\delta_{i+1}+2\gamma_i\right)\\ &= -4\sum_{i=0}^{k+1}\beta_{k,i}(\zeta_{i,2}-\zeta_{i,1})+2\sum_{i=0}^{k+1}\beta_{k,i}\gamma_i.\end{aligned}$$

Note lemma 5, and the proof is completed. □

Proposition 3. *Given an integer $m > 0$, let $\beta_{k,i}$ be defined by (3.30), and set*

$$\begin{aligned}\Phi_{2k-1} &= (-1)^k 2^{-2k-1}\left[-\frac{1}{2}\partial P_k+2\sum_{i=0}^{k+1}\beta_{k,i}(\Psi_{i,1,2k-1}-\Psi_{i,2,2k-1})\right]\\ \Phi_{2k} &= (-1)^k 2^{-2k-1}\left[\frac{1}{2}P_{k+1}-\left(\frac{1}{2}\partial^2+\frac{3}{2}q\right)\frac{1}{2}P_k-\partial\sum_{i=0}^{k+1}\beta_{k,i}(\zeta_{i,2}\Phi_{2k-1}+\Psi_{i,2,2k-1})\right.\\ &\qquad\left.+2\sum_{i=0}^{k+1}\beta_{k,i}(\Psi_{i,1,2k}-\Psi_{i,2,2k})\right] \end{aligned}\tag{3.32}$$

for $k = 1, 2, \ldots, m-1$. Then under relation (1.1) with $L = 2m-2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32) we have

$$\tilde{P}_m \equiv \sum_{i=0}^{m+1}\beta_{m,i}K_i = \frac{1}{2}P_m h^{2m}\begin{pmatrix}1\\1\end{pmatrix}+O(h^{2m+1})\tag{3.33}$$

and

$$\begin{pmatrix}w\\v\end{pmatrix}_{t_m}+\frac{1}{h^{2m-1}}J\tilde{P}_m = \frac{1}{2}\big(q_{t_m}-B_0P_m\big)h^2\begin{pmatrix}1\\1\end{pmatrix}+O(h^3).\tag{3.34}$$

Proof. It is easy to check the case when $m = 1$. If equation (3.33) is valid for m, then we have (according to lemma 4)

$$T\tilde{P}_m = \tilde{J}G\sum_{i=0}^{m+1}\beta_{m,i}K_i\tag{3.35}$$

$$\begin{aligned}&= \tilde{J}G\left[\frac{1}{2}P_m h^{2m}\begin{pmatrix}1\\1\end{pmatrix}+h^{2m+1}\sum_{i=0}^{m+1}\beta_{m,i}\begin{pmatrix}\zeta_{i,1}\Phi_{2m-1}+\Psi_{i,1,2m-1}\\ \zeta_{i,2}\Phi_{2m-1}+\Psi_{i,2,2m-1}\end{pmatrix}\right.\\ &\quad\left.+\,h^{2m+2}\sum_{i=0}^{m+1}\beta_{m,i}\begin{pmatrix}\zeta_{i,1}\Phi_{2m}+\Psi_{i,1,2m}\\ \zeta_{i,2}\Phi_{2m}+\Psi_{i,2,2m}\end{pmatrix}+\,O(h^{2m+3})\right].\end{aligned}\tag{3.36}$$

Noting the definition of Φ_{2m-1} and Φ_{2m} in (3.32), we obtain (due to (3.31))

$$-2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,1}\Phi_{2m-1}+\Psi_{i,1,2m-1})+2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,2}\Phi_{2m-1}+\Psi_{i,2,2m-1})+\frac{1}{2}\partial P_m = 0\tag{3.37}$$

and

$$\left(\frac{1}{2}\partial^2+\frac{3}{2}q\right)\frac{1}{2}P_m+\partial\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,2}\Phi_{2m-1}+\Psi_{i,2,2m-1})-2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,1}\Phi_{2m}+\Psi_{i,1,2m})$$
$$+2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,2}\Phi_{2m}+\Psi_{i,2,2m})=\frac{1}{2}P_{m+1}. \qquad (3.38)$$

Combining the above two equations (3.37) and (3.38), and noting equation (2.17), we have

$$\left(\frac{1}{2}\partial^2-\frac{1}{2}\partial^{-1}q_x+\frac{1}{2}\partial^{-1}q\partial\right)\frac{1}{2}P_m-\partial\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,1}\Phi_{2m-1}+\Psi_{i,1,2m-1})$$
$$+2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,1}\Phi_{2m}+\Psi_{i,1,2m})-2\sum_{i=0}^{m+1}\beta_{m,i}(\zeta_{i,2}\Phi_{2m}+\Psi_{i,2,2m})=\frac{1}{2}P_{m+1}. \qquad (3.39)$$

So we get

$$T\tilde{P}_m=\frac{1}{2}P_{m+1}h^{2m+2}\begin{pmatrix}1\\1\end{pmatrix}+O(h^{2m+3}). \qquad (3.40)$$

On the other hand (according to lemma 2),

$$T\tilde{P}_m=\tilde{J}G\sum_{i=0}^{m+1}\beta_{m,i}K_i=\sum_{i=0}^{m+1}\beta_{m,i}(K_{i+1}+\delta_{i+1}K_0)=\tilde{P}_{m+1}. \qquad (3.41)$$

Equation (3.34) is the corollary of equation (3.33) and proposition 2. The proof is completed. □

We give an example here. For $m=3$, using proposition 3, we can get

$$\Phi_1=\tfrac{1}{8}q_x \qquad \Phi_2=-\tfrac{1}{32}q^2 \qquad \Phi_3=-\tfrac{1}{384}q_{xxx} \qquad \Phi_4=\tfrac{1}{254}\left(q^3+qq_{xx}+q_x^2\right) \qquad (3.42)$$

then under relation (1.1) with $L=4$, $f(x,t)=\frac{1}{2}q(x,t)$ and the above Φ_i we have

$$-10K_0+4K_1-2K_2+2K_3+K_4=\frac{1}{2}P_3h^6\begin{pmatrix}1\\1\end{pmatrix}+O(h^7).$$

In the previous work in [11], we must combine $K_0, K_1, \ldots, K_6$ to give P_3 under relation (1.1) with $L=0$. In general, $K_0, K_1, \ldots, K_{2m}$ need to be combined to give P_m under relation (1.1) with $L=0$ [11]. Proposition 3 shows us that almost half of them, i.e. $K_0, K_1, \ldots, K_{m+1}$, are needed to give P_m by introducing $\Phi_i(f)$ (3.32). Furthermore, according to the recursion formula for $\Phi_i(f)$ (3.32) it is easy to see that all the $\Phi_i(f)$, introduced by (3.32), are differential polynomials of f, and our process for finding $\Phi_i(f)$ can be continued indefinitely.

In what follows, we will derive the continuous limit relations between the Hamiltonians, the Lax pairs of the Toda hierarchy and those of the KdV hierarchy, respectively.

Lemma 7. *If there is a relation between $\tilde{w}(n)$, $n\in\mathbb{Z}$, and $q(x)$, $x\in\mathbb{R}$*

$$\tilde{w}(n)=q^{(s_1)}(x)q^{(s_2)}(x)\cdots q^{(s_m)}(x)h^l \qquad (3.43)$$

where h is the step of the lattice, $x=nh$, s_i, $1\leqslant i\leqslant m$ and l are non-negative integers, and denoting $\tilde{S}$ as the operator which stands for submitting relation (3.43) into a polynomial of $\tilde{w}, \tilde{w}^{(-1)}, \tilde{w}^{(1)}, \ldots$, and then expanding in Taylor series, we have the formula

$$\frac{\delta}{\delta q}\circ\tilde{S}=h^l\tilde{Z}\circ\tilde{S}\circ\frac{\delta}{\delta\tilde{w}} \qquad (3.44)$$

where $\tilde{Z}$ stands for a differential operator.

The proof for lemma 7 is given in appendix A.

Proposition 4. *Given an integer $m > 0$, set*

$$\tilde{H}_m \equiv \sum_{i=0}^{m+1} \beta_{m,i} H_i - \sum_{i=1}^{m+1} \beta_{m,i} \frac{\alpha_{i+1}}{i} \tag{3.45}$$

under relation (1.1) with $L = 2m - 2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32), we have

$$\int S(\tilde{H}_m)\,\mathrm{d}x = \frac{1}{2}h^{2m+2}\int \bar{H}_m\,\mathrm{d}x + O(h^{2m+3}) \tag{3.46}$$

where S is an operator which stands for submitting relation (1.1) with $L = 2m - 2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32) into a polynomial of $w, v, w^{(-1)}, v^{(-1)}, w^{(1)}, v^{(1)}, \ldots,$ and then expanding in Taylor series.

Proof. According to lemma 7, under relation (1.1) with $L = 2m - 2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32) (since Φ_i are differential polynomials of q), we have

$$\begin{aligned}
\frac{\delta}{\delta q}\circ S &= \sum_{j=0}^{\infty}(-\partial)^j \frac{\partial}{\partial q^{(j)}}\circ S \\
&= \sum_{j=0}^{\infty}(-\partial)^j \sum_{k\in\mathbb{Z}}\left[\left(\frac{\partial S\big(w^{(k)}\big)}{\partial q^{(j)}}\right) S\circ\frac{\partial}{\partial w^{(k)}} + \left(\frac{\partial S\big(v^{(k)}\big)}{\partial q^{(j)}}\right) S\circ\frac{\partial}{\partial v^{(k)}}\right] \\
&= \frac{1}{2}h^2\sum_{j=0}^{\infty}(-\partial)^j\sum_{k\in\mathbb{Z}}\frac{(kh)^j}{j!}S\circ\left(\frac{\partial}{\partial w^{(k)}} + \frac{\partial}{\partial v^{(k)}}\right) + h^3 Z\circ S\circ\left(\frac{\delta}{\delta w} - \frac{\delta}{\delta v}\right) \\
&= \frac{1}{2}h^2 S\circ\left(\frac{\delta}{\delta w} + \frac{\delta}{\delta v}\right) + h^3 Z\circ S\circ\left(\frac{\delta}{\delta w} - \frac{\delta}{\delta v}\right)
\end{aligned}$$

where Z stands for a differential operator, and we do not care about its concrete form. Note lemma 1 and the definition of H_i in (2.10), we can have the expansion

$$S(\tilde{H}_m) = \sum_{i=2}^{\infty}\tilde{H}_{m,i}h^i$$

where $\tilde{H}_{m,i}|_{q=0} = 0$, and according to proposition 3, we have

$$\begin{aligned}
\frac{\delta}{\delta q}\circ S(\tilde{H}_m) &= \sum_{i=2}^{\infty} h^i\frac{\delta\tilde{H}_{m,i}}{\delta q} \\
&= \left[\frac{1}{2}h^2 S\circ\left(\frac{\delta}{\delta w} + \frac{\delta}{\delta v}\right) + h^3 Z\circ S\circ\left(\frac{\delta}{\delta w} - \frac{\delta}{\delta v}\right)\right]\sum_{i=0}^{m+1}\beta_{m,i}H_m \\
&= \frac{1}{2}h^{2m+2}\frac{\delta\bar{H}_m}{\delta q} + O(h^{2m+3}).
\end{aligned}$$

Then one can get [12]

$$\tilde{H}_{m,i} \in \text{const} + \text{image}(\partial) \qquad 2 \leqslant i \leqslant 2m+1.$$

As we mentioned above, there is no constant item in each $\tilde{H}_{m,i}$, $i \geqslant 2$ (i.e. $\tilde{H}_{m,i}|_{q=0} = 0$), so

$$\int \tilde{H}_{m,i}\,\mathrm{d}x = 0 \qquad 2 \leqslant i \leqslant 2m+1.$$

Just using the same deduction, we conclude

$$\int \tilde{H}_{m,2m+2}\,\mathrm{d}x = \frac{1}{2}\int \bar{H}_m\,\mathrm{d}x$$

which completes the proof. □

Lemma 8. *Under relation (1.1) with* $f(x,t) = \frac{1}{2}q(x,t)$*, we have*

$$A_k = \alpha_k - \gamma_k + \sum_{i=2}^{\infty} A_{k,i}h^i \qquad k = 0, 1, \ldots \tag{3.47}$$

where

$$A_{k,2i}|_{q=0} = 0 \qquad A_{k,2i+1}|_{q=0} = \xi_{k,2i+1}\partial^{2i+1} \qquad i = 1, 2, \ldots \tag{3.48}$$

$\xi_{k,2i+1}$ *is a constant, and* α_k *and* γ_k *are given in lemma 1.*

Proof. For $k = 0$ and $k = 1$, we have

$$A_0|_{q=0} = -\frac{1}{2} \qquad A_1|_{q=0} = 1 + \sum_{j=0}^{\infty}\frac{1}{(2j+1)!}h^{2j+1}(-\partial)^{2j+1}.$$

If the lemma is valid for $k-1$, note $\alpha_k = -2\alpha_{k-1} + 2\gamma_{k-1}$ (see lemma 1), we have

$$\begin{aligned}
A_k|_{q=0} &= A_{k-1}\left(E + w + vE^{(-1)}\right) - vb_k^{(1)}E^{(-1)} - a_k|_{q=0}\\
&= \left[\alpha_{k-1} - \gamma_{k-1} + \sum_{i=0}^{\infty}\xi_{k-1,2i+1}h^{2i+1}\partial^{2i+1}\right]\sum_{j=1}^{\infty}\frac{2}{(2j)!}h^{2j}\partial^{2j} + \alpha_k\sum_{j=0}^{\infty}\frac{1}{j!}h^j(-\partial)^j - \gamma_k\\
&\equiv \alpha_k - \gamma_k + \sum_{i=0}^{\infty}\xi_{k,2i+1}h^{2i+1}\partial^{2i+1}.
\end{aligned}$$

□

Lemma 9. *Define*

$$\tilde{A}_k \equiv \sum_{i=1}^{k+1}\beta_{k,i}A_{i-1} \qquad k = 1, 2 \ldots. \tag{3.49}$$

Then under relation (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$*, we have*

$$\tilde{A}_k = \sum_{i=2}^{\infty}\tilde{A}_{k,i}h^i \tag{3.50}$$

where

$$\tilde{A}_{k,2i}|_{q=0} = 0 \qquad \tilde{A}_{k,2i+1}|_{q=0} = \tilde{\xi}_{k,2i+1}\partial^{2i+1} \qquad i = 1, 2, \ldots \tag{3.51}$$

$\tilde{\xi}_{k,2i+1}$ *is a constant.*

Proof. According to lemma 8, we only need to prove

$$\sum_{i=1}^{k+1}\beta_{k,i}(\alpha_{i-1} - \gamma_{i-1}) = 0. \tag{3.52}$$

It is easy to check the cases: $k = 1$ and $k = 2$, and for $k \geqslant 3$, note lemma 5, we have

$$\sum_{i=1}^{k+1}\beta_{k,i}(\alpha_{i-1} - \gamma_{i-1}) = \sum_{i=1}^{k+1}\beta_{k-1,i-1}(\alpha_{i-1} - \gamma_{i-1}) = \sum_{i=0}^{k}\beta_{k-1,i}(\alpha_i - \gamma_i) = 0$$

which completes the proof. □

Proposition 5. *Given an integer $m > 0$, under relation (1.1) with $L = 2m - 2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32), we have*

$$\tilde{A}_m \equiv \sum_{i=1}^{m+1} \beta_{m,i} A_{i-1} = -\bar{A}_m h^{2m-1} + O(h^{2m}). \tag{3.53}$$

Proof. It is valid for $m = 1, 2$. According to proposition 3, we have

$$\begin{aligned}
[\tilde{A}_m, L] &= \sum_{i=1}^{m+1} \beta_{m,i} \frac{\mathrm{d}w}{\mathrm{d}t_{i-1}} + \sum_{i=1}^{m+1} \beta_{m,i} \frac{\mathrm{d}v}{\mathrm{d}t_{i-1}} E^{(-1)} \\
&= J_{12} \sum_{i=1}^{m+1} \beta_{m,i} K_{i,2} + J_{21} \sum_{i=1}^{m+1} \beta_{m,i} K_{i,1} E^{(-1)} \\
&= -B_0 P_m h^{2m+1} + O(h^{2m+2}) \\
&= -[\bar{A}_m, \bar{L}] h^{2m+1} + O(h^{2m+2}).
\end{aligned} \tag{3.54}$$

Under relation (1.1) with $L = 2m - 2$, $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32), proposition 1 and lemma 9 together imply

$$L = \bar{L}h^2 + \sum_{i=3}^{\infty} L_i h^i \qquad \tilde{A}_m = \sum_{i=2}^{\infty} \tilde{A}_{m,i} h^i \tag{3.55}$$

where L_i and $\tilde{A}_{m,i}$ are differential operators. Comparing the terms of h^4 in (3.54), we know

$$[\tilde{A}_{m,2}, \bar{L}] = 0. \tag{3.56}$$

According to [16], $\tilde{A}_{m,2}$ can be represented by

$$\tilde{A}_{m,2} = \sum_{j=0}^{\infty} \eta_{m,2,j} (\bar{L})^j \tag{3.57}$$

where $\eta_{m,2,j}$ are constants. Noting lemma 9, we have

$$\tilde{A}_{m,2}|_{q=0} = 0 = \sum_{j=0}^{\infty} \eta_{m,2,j} (\partial^2)^j. \tag{3.58}$$

Then one can get $\eta_{m,2,j} = 0$ for all j, and

$$\tilde{A}_{m,2} = 0. \tag{3.59}$$

Comparing the terms of h^5 in (3.54), we know

$$[\tilde{A}_{m,3}, \bar{L}] = 0 \tag{3.60}$$

then $\tilde{A}_{m,3}$ can be represented by [16]

$$\tilde{A}_{m,3} = \sum_{j=0}^{\infty} \eta_{m,3,j} (\bar{L})^j \tag{3.61}$$

where $\eta_{m,3,j}$ are constants. Note lemma 9, and we have

$$\tilde{A}_{m,3}|_{q=0} = \tilde{\xi}_{m,3} \partial^3 = \sum_{j=0}^{\infty} \eta_{m,3,j} (\partial^2)^j. \tag{3.62}$$

Then one can get $\eta_{m,3,j} = 0$ for all j, and

$$\tilde{A}_{m,3} = 0. \tag{3.63}$$

In the same way, we conclude

$$\tilde{A}_{m,i} = 0 \qquad i = 2, \ldots, 2m-2. \tag{3.64}$$

Comparing the terms of h^{2m+1} in (3.54), we know

$$[\tilde{A}_{m,2m-1}, \bar{L}] = -[\bar{A}_m, \bar{L}] \tag{3.65}$$

then $\tilde{A}_{m,2m-1} + \bar{A}_m$ can be represented by [16]

$$\tilde{A}_{m,2m-1} + \bar{A}_m = \sum_{j=0}^{\infty} \eta_{m,2m-1,j} (\bar{L})^j \tag{3.66}$$

where $\eta_{m,2m-1,j}$ are constants. Noting lemma 9 and (2.20), we have

$$(\tilde{A}_{m,2m-1} + \bar{A}_m)|_{q=0} = \tilde{\xi}_{m,2m-1} \partial^{2m-1} + \partial^{2m-1} = \sum_{j=0}^{\infty} \eta_{m,2m-1,j} (\partial^2)^j. \tag{3.67}$$

Then we get $\eta_{m,2m-1,j} = 0$ for all j and

$$\tilde{A}_m \equiv \sum_{i=1}^{2m} \beta_{m,i} A_{i-1} = -\bar{A}_m h^{2m-1} + O(h^{2m}). \tag{3.68}$$

Thus the proof is completed. □

4. Conclusions and remarks

In this paper, by introducing the higher order terms in the potential expansion, we have proved that there is a continuous limit relation between the Toda hierarchy and the KdV hierarchy. Compared with [11], fewer members of the Toda hierarchy are needed to recover the KdV hierarchy by the recombination method. For example, proposition 3 shows that under the potential expansion (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$ and (3.32), we can combine $K_0, K_1, \ldots, K_{m+1}$, to get P_m in the continuous limit. However, under the lower finite potential expansion, for example (1.1) with $f(x,t) = \frac{1}{2}q(x,t)$ and $L = 0$, we need $K_0, K_1, \ldots, K_m, \ldots, K_{2m}$, to recover P_m through the continuous limit process [11].

Compared with [10], a new method of introducing $\Phi_i(f)$ in the potential expansion (1.1) was presented in this paper. Moreover, from the recursion formula for $\Phi_i(f)$ (3.32), it is easy to see that the $\Phi_i(f)$, introduced in our construction, are all differential polynomials of f, and our process for determining $\Phi_i(f)$ can be continued indefinitely. However, this cannot be obtained in [10], since the $\Phi_i(f)$ are obtained by integration there.

It was also shown that the Lax pairs, the Poisson tensors and the Hamiltonians of the Toda hierarchy tend towards the corresponding ones of the KdV hierarchy in the continuous limit.

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Appendix. Proof of lemma 7

Denote $\tilde{w}_i = q^{(s_1)} \cdots q^{(s_{i-1})} q^{(s_{i+1})} \cdots q^{(s_m)}$, for $i = 1, \ldots, m$, then we have

$$
\begin{aligned}
\frac{\delta}{\delta q} \circ \tilde{S} &= \sum_{j=0}^{\infty} (-\partial)^j \frac{\partial}{\partial q^{(j)}} \circ \tilde{S} \\
&= \sum_{j=0}^{\infty} (-\partial)^j \sum_{k\in\mathbb{Z}} \left(\frac{\partial \tilde{S}(\tilde{w}^{(k)})}{\partial q^{(j)}} \right) \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} \sum_{j=s_i}^{\infty} (-\partial)^j \sum_{k\in\mathbb{Z}} \frac{(kh)^{j-s_i}}{(j-s_i)!} \left(\mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} \sum_{j=0}^{\infty} (-\partial)^{j+s_i} \sum_{k\in\mathbb{Z}} \frac{(kh)^{j}}{j!} \left(\mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{p=0}^{j} \frac{(-kh)^j}{p!(j-p)!} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \partial^{j-p} \circ \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{p=0}^{\infty} \sum_{k\in\mathbb{Z}} \sum_{j=p}^{\infty} \frac{(-kh)^j}{p!(j-p)!} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \partial^{j-p} \circ \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{p=0}^{\infty} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \frac{(-kh)^{j+p}}{p!j!} \partial^{j} \circ \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{p=0}^{\infty} \frac{(-kh)^p}{p!} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \frac{(-kh)^{j}}{j!} \partial^{j} \circ \tilde{S} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{p=0}^{\infty} \frac{(-kh)^p}{p!} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \tilde{S} \circ \sum_{k\in\mathbb{Z}} E^{(-k)} \circ \frac{\partial}{\partial \tilde{w}^{(k)}} \\
&= h^l \sum_{i=1}^{m} (-\partial)^{s_i} \sum_{p=0}^{\infty} \frac{(-kh)^p}{p!} \left(\partial^p \mathrm{e}^{kh\partial} \tilde{S}(\tilde{w}_i) \right) \tilde{S} \circ \frac{\delta}{\delta \tilde{w}^{(k)}} \\
&\equiv h^l \tilde{Z} \circ \tilde{S} \circ \frac{\delta}{\delta \tilde{w}}.
\end{aligned}
$$

The proof for lemma 7 is completed. □

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